Figure 3 shows the effect of angle of attack on the pressure drag coefficient at a given Reynolds number. Again, the pressure jump on the surface is included. It is seen that the effect of the angle of attack is to reduce the pressure drag for a given body. The reduction in the drag increases as the bluntness of the body increases.

In Fig. 4, the heating rate results are plotted with and without slip conditions at $\alpha = 0°$ and $8°$. The effect of slip is to reduce the heating rates. The magnitude of the difference is maximum in the nose region of the body and decreases for the downstream points.

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Technical Comments

Comment on "Discretized Simulation of Vortex Sheet Evolution with Buoyancy and Surface Tension Effects"

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ZALOSH[1] carried out calculations for the roll-up of a vortex sheet separating inviscid fluids of different densities. The derivation of his Eq. (14) for the rate of change of circulation following a discrete vortex contains an error that results in the omission of a term from the equation. For a continuous vortex sheet without surface tension effects, the correct expression for the rate of change of circulation per unit length was given in an earlier paper by Zaroodny and Greenberg.[2] Zalosh's numerical results probably would be little changed by the use of the correct equation, because his calculations were for small density differences, and the omitted term is proportional to the density difference across the vortex sheet. On the other hand, use of Zalosh's equation for a problem involving large density differences would result in large errors.

The error of Ref. 1 involves the convective part of the total (Lagrangian) derivative. Proper care was not taken in distinguishing between derivatives following fluid particles on either side of the vortex sheet and the derivative following the vortex motion. The rate of change of circulation following a closed circuit C moving with the fluid is given by[3]

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{(C)} \boldsymbol{V}\cdot \mathrm{d}\boldsymbol{r} = \int_{(C)} \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\cdot \mathrm{d}\boldsymbol{r} + \int_{(C)} \boldsymbol{V}\cdot \mathrm{d}\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right) \tag{1}$$

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Because the Lagrangian derivative follows the fluid particles, $\mathrm{d}\boldsymbol{r}/\mathrm{d}t = \boldsymbol{V}$, and the last integral in Eq. (1) vanishes. Consider a closed circuit centered at a position s_i on a vortex sheet, and having sides of length ℓ just above and below the sheet. The rate of change of (clockwise) circulation around the vortex sheet segment is

$$\frac{\mathrm{d}\Gamma_i}{\mathrm{d}t} = \int_{s_i-\ell/2}^{s_i+\ell/2} \left[\left(\frac{\mathrm{d}\boldsymbol{V}_1}{\mathrm{d}t}\right)_s - \left(\frac{\mathrm{d}\boldsymbol{V}_2}{\mathrm{d}t}\right)_s\right]\mathrm{d}s \tag{2}$$

where $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ denote the velocities just above and below the vortex sheet, the subscript s denotes the components tangential to the vortex sheet, and $\mathrm{d}s$ is the differential distance along the sheet. Equation (2) has the same form as Eq. (7) of Ref. 1, provided that we interpret Zalosh's terms $\mathrm{d}V_{1s}/\mathrm{d}t$ and $\mathrm{d}V_{2s}/\mathrm{d}t$ as the tangential components of the accelerations, i.e., as $(\mathrm{d}\boldsymbol{V}_1/\mathrm{d}t)_s$ and $(\mathrm{d}\boldsymbol{V}_2/\mathrm{d}t)_s$, respectively. However, contrary to the statement by Zalosh, the Lagrangian derivatives follow the fluid particles, not the ith vortex; i.e.,

$$(\mathrm{d}\boldsymbol{V}_1/\mathrm{d}t)_s \equiv (\partial \boldsymbol{V}_1/\partial t)_s + (\boldsymbol{V}_1\cdot\nabla \boldsymbol{V}_1)_s$$

$$(\mathrm{d}\boldsymbol{V}_2/\mathrm{d}t)_s \equiv (\partial \boldsymbol{V}_2/\partial t)_s + (\boldsymbol{V}_2\cdot\nabla \boldsymbol{V}_2)_s$$

The momentum equation [Eq. (8) in Ref. 1] is used to determine the acceleration terms in Eq. (2)

$$\left(\frac{\mathrm{d}\boldsymbol{V}_1}{\mathrm{d}t}\right)_s = -\frac{1}{\rho_1}\frac{\partial p_1}{\partial s} - g\sin\theta, \quad \left(\frac{\mathrm{d}\boldsymbol{V}_2}{\mathrm{d}t}\right)_s = -\frac{1}{\rho_2}\frac{\partial p_2}{\partial s} - g\sin\theta \tag{3}$$

where again the total derivatives follow fluid particles in the upper and lower fluids, respectively. Substitution of Eqs. (3) into Eq. (2) yields Zalosh's Eq. (9), which is correct.

The velocity of a vortex is given as the average of the fluid velocities above and below; i.e., $\boldsymbol{V}_i = \frac{1}{2}(\boldsymbol{V}_1 + \boldsymbol{V}_2)$. Differentiation following the vortex yields an equation of the form of Eq. (12) of Ref. 1:

$$\left(\frac{\mathrm{d}\boldsymbol{V}_i}{\mathrm{d}t}\right)_s = \frac{1}{2}\left[\left(\frac{\mathrm{d}\boldsymbol{V}_1}{\mathrm{d}t}\right)_s + \left(\frac{\mathrm{d}\boldsymbol{V}_2}{\mathrm{d}t}\right)_s\right] \tag{4}$$

where now all of the Lagrangian time derivatives in Eq. (4) are evaluated with respect to the vortex velocity; i.e., $d(\ )/dt = \partial(\ )/\partial t + \frac{1}{2}(V_1 + V_2)\cdot\nabla(\ )$ for each derivative in Eq. (4). Thus the derivatives on the right-hand side of Eq. (4) are different from the derivatives on the left-hand sides of Eqs. (3), and Eqs. (3) may not be substituted directly into Eq. (4), as was done in Ref. 1.

By properly accounting for the differences in the Lagrangian derivatives following the vortex in Eq. (4) and the Lagrangian derivatives following the fluid particles in Eqs. (3), the final equation for the rate of change of circulation becomes

$$\frac{d\Gamma_i}{dt} = 2\kappa\Delta s_i\left[g\sin\theta_i + \left(\frac{dV_i}{dt}\right)_s + \frac{1}{4}\gamma_i\left(\frac{\partial\gamma}{\partial s}\right)_{s=s_i}\right] - \frac{2\sigma}{\rho_1+\rho_2}\Delta s_i\left(\frac{\partial R^{-1}}{\partial s}\right)_{s=s_i} \tag{5}$$

where $\kappa \equiv (\rho_2-\rho_1)/(\rho_1+\rho_2)$, and $\gamma_i = (V_{1s}-V_{2s})_{s=si}$ is the circulation per unit arc length [the approximation $\gamma_i = \Gamma_i/\Delta s_i$ can be used for numerical calculations, with an appropriate finite-differencing for the $(\partial\gamma/\partial s)_{s=si}$ factor]. Equation (5) differs from Eq. (14) of Ref. 1 by the addition of the $\frac{1}{4}\gamma_i(\partial\gamma/\partial s)_{s=s_i}$ term in the brackets. Equation (5) is consistent with the equation given by Zaroodny and Greenberg[2] for a continuous vortex sheet.

In order to determine the magnitude of the error incurred by using Eq. (14) of Ref. 1, we now consider the linearized case corresponding to the classical Kelvin-Helmholtz problem. For a coordinate system at rest with respect to an unperturbed vortex sheet along the x direction, linearization is achieved by first setting $\gamma = \gamma_0 + \gamma'$, $V_x = V_x'$, $V_y = V_y' = \partial\eta/\partial t$, with γ being the circulation per unit length, γ_0 is the unperturbed circulation per unit length (which is equal to the unperturbed velocity change ΔU across the sheet), and η is the interface displacement. After linearization, the continuous form of Eq. (5) [Eq. (12) of Ref. 2] and the Biot-Savart integrals for the vortex velocity components become

$$\frac{\partial\gamma'}{\partial t} = -\gamma_0\frac{\partial V_x'}{\partial x} + 2\kappa\left[g\frac{\partial\eta}{\partial x} + \frac{\partial V_x'}{\partial t} + \frac{1}{4}\gamma_0\frac{\partial\gamma'}{\partial x}\right] - \frac{2\sigma}{\rho_1+\rho_2}\frac{\partial^3\eta}{\partial x^3} \tag{6a}$$

$$V_x' = \frac{\gamma_0}{2\pi}\int_{-\infty}^{\infty}\frac{[\eta(x,t)-\eta(\xi,t)]}{(x-\xi)^2}d\xi \tag{6b}$$

$$\frac{\partial\eta}{\partial t} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\gamma'(\xi,t)}{(x-\xi)}d\xi \tag{6c}$$

where the integrals are determined by their Cauchy principal values.

If γ', V_x', and η are assumed to be proportional to $\exp(ikx+nt)$, then Eqs. (6) are solved by the following value of n;

$$n = i\frac{\kappa}{2}\Delta Uk \pm\left[\frac{1}{4}(1-\kappa^2)(\Delta U)^2k^2 - \kappa gk - \frac{\sigma k^3}{\rho_1+\rho_2}\right]^{1/2} \tag{7}$$

Equation (7) is the classical Kelvin-Helmholtz result, and is given as Eq. (1) in Ref. 1. If Eq. (14) of Ref. 1 is used for the rate of change of circulation, then the linearized calculation gives an expression for n of the same form as Eq. (7), but with κ replaced by $(\frac{1}{2})\kappa$ in the first term on the right, and κ^2 by $(\frac{1}{4})\kappa^2$ in the first term under the square root.

Although Zalosh's numerical calculations apparently were matched to the correct linear Kelvin-Helmholtz formula at the initial time, errors in his use of his Eq. (14) should be comparable to the differences in the two linear values for n. For the small density differences considered by Zalosh, $\kappa < 0.053$, the two equations for n give little difference in the periods of amplitude oscillation for stable conditions. The two formulas for n give very small differences in the amplitude growth rates for the unstable conditions considered by Zalosh, and the phase velocities are small (but differ by a factor of two). Thus, we would expect his numerical results to be reasonably accurate. On the other hand, if Eq. (14) of Ref. 1 is used for large density differences, i.e., κ not small, then large errors result. Note that, in the single-fluid limit of $\rho_1 \to 0$, i.e., $\kappa \to 1$, Eq. (7) shows that the shear instability term $(\frac{1}{4})(1-\kappa^2)(\Delta U)^2$ disappears. For the incorrect formula derived from Eq. (14) of Ref. 1, the shear instability would remain in the limit $\kappa \to 1$.

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THE comments on the missing term in the circulation equation are painfully acknowledged. In general, the term

$$1/4\ \gamma_i(\partial\gamma/\partial s)_{s=s_i}$$

should appear in the brackets in Eq. (14) of Ref. 1, and the term

$$\frac{1}{4}\frac{(1-S)}{(1+S)}\gamma_i(\gamma_{i+1}-\gamma_{i-1})$$

should appear on the right-hand side of Eq. (16c).

Fortunately, the addition of the missing term should not alter significantly the calculated results presented in Ref. 1. This can be seen by evaluating the ratio of the missing term to the gravitational acceleration term, i.e., the first term on the right-hand side of Eqs. (14) and (16c). For the initial sinusoidally perturbed vortex sheet, this ratio (rms averaged over one wavelength) is equal to

$$\left|\frac{\overline{\dfrac{\Delta U}{4}\dfrac{\partial\gamma}{\partial s}}}{\overline{g\dfrac{\partial\eta}{\partial s}}}\right| = \frac{\pi}{2}\frac{(\Delta U)^2}{g\lambda}$$

Most of the calculated results presented in Ref. 1 were obtained for small values of the ratio $\pi(\Delta U)^2/2g\lambda$. For example, the curves shown in Fig. 6 of Ref. 1 ($F_r^{-2}=1$, $S=0.9$) correspond to a ratio of 0.083. The only case for which this ratio is not much smaller than unity is the situation depicted in Fig. 7 of Ref. 1 ($F_r^{-2}=0.10$, $S=0.90$). In that case, the 7% discrepancy between the calculated period of oscillation and the results of linear theory, may be due partially to the inadvertently omitted term in the circulation equation.

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